Game with asymmetric information

An employer must hire an employee for the firm with the following profit function $\pi = ke - w$. Where w is the wage, and e is the effort provided by the employee. Potential employees can be of two types, *Good* or *Bad*, with the following utility functions:

$$U^G(w, e) = w - e^2$$

$$U^B(w, e) = w - 2e^2$$

1. Solve the employer's problem with perfect information about the type of employee. Find the wage and effort demanded for each type.
2. Solve the employer's problem when there is imperfect information.
3. Compare the two cases above.

Solution

1. In the case where the employer knows which type of worker they are dealing with, they will maximize the following function:

$$\pi(e) = ke - w$$

At the same time, the following two conditions must be met for individuals of both types to accept the job:

$$w - e^2 \geq \text{ Reservation wage}$$

$$w - 2e^2 \geq \text{ Reservation wage}$$

In this case:

$$w - e^2 \geq 0$$

$$w - 2e^2 \geq 0$$

As the employer's goal is to maximize their profit, both equations will be satisfied with equality:

$$w - e^2 = 0$$

$$w - 2e^2 = 0$$

From these two equations, we have that $e^2 = w$ and $2e^2 = w$ for the Good and Bad types respectively. So, for the good type, the goal would be to maximize the following profit function:

$$\pi(e) = ke - w = ke - e^2$$

The first-order condition for maximization:

$$\frac{\partial \pi}{\partial e} = k - 2e = 0$$

Therefore, $e = \frac{k}{2}$. And the wage in this case is: $w = e^2 = (k/2)^2$ For the other type, the profit function is:

$$\pi(e) = ke - w = ke - 2e^2$$

Maximizing:

$$\frac{\partial \pi}{\partial e} = k - 4e = 0$$

Then, the effort demanded will be: $e = \frac{k}{4}$ and the wage $w = 2e^2 = 2(k/4)^2$.

2. In the case where the employer does not know which type of individual they are facing, they have to make a demand for effort and offer a wage that considers both types of individuals. The function to maximize should be the following:

$$\pi = q(ke_G - w_G) + (1 - q)(ke_B - w_B)$$

But as before, it faces constraints. First, it has to offer a wage that compensates for the reservation wage of both types of individuals (participation constraints):

$$w_G - e_G^2 \geq 0 \tag{1}$$

$$w_B - 2e_B^2 \geq 0 \tag{2}$$

On the other hand, now two additional constraints must also be imposed. These constraints prevent individuals of the Good type from passing off as Bad type, and those of the Bad type from passing off as Good type (incentive compatibility constraints):

$$\overbrace{w_G - e_G^2}^{\text{Utility of Good type with Good type wage and effort}} \geq \overbrace{w_B - e_B^2}^{\text{Utility of Good type, with Bad type wage and effort}} \tag{3}$$

$$\overbrace{w_B - 2e_B^2}^{\text{Utility of Bad type with Bad type wage and effort}} \geq \overbrace{w_G - 2e_G^2}^{\text{Utility of Bad type, with Good type wage and effort}} \tag{4}$$

We then have 4 conditions and a function to maximize. But we can summarize this in two conditions as follows. We take (3) and at the same time, we can relate it to (2):

$$\overbrace{w_G - e_G^2 \geq w_B - e_B^2}^{\text{Condition (3)}} \geq \underbrace{w_B - 2e_B^2 \geq 0}_{\text{Condition (2)}}$$

These two conditions are related because the two terms in the middle are equal except that the one on the right is smaller since there is a 2 multiplying e_B^2. Therefore, if (2) ($w_B - 2e_B^2 \geq 0$) is fulfilled, (1) ($w_G - e_G^2 \geq 0$) must be fulfilled. We can discard (1), Also rewriting (3):

$$w_G \geq e_G^2 + w_B - e_B^2$$

This is going to be satisfied with equality since it will attempt to minimize w_G as much as possible. Lastly, note that only one of (2) and (4) can be satisfied with equality, re-expressing:

$$w_B \geq 2e_B^2$$

$$w_B \geq 2e_B^2 + w_G - 2e_G^2$$

We assume that (4) is satisfied with equality and we will reach a contradiction:

$$w_B = w_G + 2e_B^2 - 2e_G^2$$

Replacing w_G since (3) is satisfied with equality:

$$w_B = e_G^2 + w_B - e_B^2 + 2e_B^2 - 2e_G^2$$
$$e_G^2 = e_B^2$$

Which is contradictory since the Bad type individuals exert less effort than the Good type individuals. We can conclude that we only need to take into account the constraints: (2) and (3). The practical rule to remember this is that **the participation constraint of the Bad type is satisfied with equality, and the incentive compatibility constraint of the Good type is satisfied with equality.** Now, let's construct the Lagrangian:

$$L = q(ke_G - w_G) + (1-q)(ke_B - w_B) + \lambda_1(w_B - 2e_B^2) + \lambda_2(w_G - e_G^2 - w_B + e_B^2)$$

The first-order conditions with respect to each variable are:

$$\frac{\partial L}{\partial e_G} = qk - 2\lambda_2 e_G = 0$$

$$\frac{\partial L}{\partial e_B} = (1-q)k - 4\lambda_1 e_B + 2\lambda_2 e_B = 0$$

$$\frac{\partial L}{\partial w_G} = -q + \lambda_2 = 0$$

$$\frac{\partial L}{\partial w_B} = -(1-q) + \lambda_1 - \lambda_2 = 0$$

$$\frac{\partial L}{\partial \lambda_1} = w_B - 2e_B^2 = 0$$

$$\frac{\partial L}{\partial \lambda_2} = w_G - e_G^2 - w_B + e_B^2 = 0$$

From the third equation, we know that $\lambda_2 = q$. We replace this in the first and obtain:

$$qk - 2qe_G = 0$$

Then:

$$\frac{k}{2} = e_G$$

With the fourth equation, we get the value of λ_1

$$-1 + q + \lambda_1 - q = 0$$

$$\lambda_1 = 1$$

With this, we go to the second equation to obtain e_B:

$$(1-q)k - 41e_B + 2qe_B = 0$$

$$\frac{k(1-q)}{(4-2q)} = e_B$$

That is: $e_B = \dfrac{k\,(1-q)}{2\,(2-q)}$ We go to the fifth equation:

$$w_B - 2\left(\frac{k\,(1-q)}{2\,(2-q)}\right)^2 = 0$$

Therefore:

$$w_B = 2\left(\frac{k\,(1-q)}{2\,(2-q)}\right)^2$$

Finally, with the sixth equation:

$$w_G - \left(\frac{k}{2}\right)^2 - 2\left(\frac{k\,(1-q)}{2\,(2-q)}\right)^2 + \left(\frac{k\,(1-q)}{2\,(2-q)}\right)^2 = 0$$

$$w_G = \left(\frac{k}{2}\right)^2 + \left(\frac{k\,(1-q)}{2\,(2-q)}\right)^2$$

The payment for the Bad type employee leaves him indifferent to his reservation price:

$$u_B = w_B - 2e_B^2 = 2\left(\frac{k\,(1-q)}{2\,(2-q)}\right)^2 - 2\left(\frac{k\,(1-q)}{2\,(2-q)}\right)^2 = 0$$

Whereas the Good type agent has his reservation price plus an extra:

$$u_G = w_G - e_G^2 = \left(\frac{k}{2}\right)^2 + \left(\frac{k\,(1-q)}{2\,(2-q)}\right)^2 - \left(\frac{k}{2}\right)^2 = \left(\frac{k\,(1-q)}{2\,(2-q)}\right)^2 > 0$$

This is called information rent, as it is the additional amount that must be paid to the Bad type employee so that he reveals his type (i.e., he complies with the incentive compatibility constraint).

3. Comparing the two situations, the situation of the Bad type does not change as he has the same utility (0). For the Good type individual, his utility is now higher due to the information rent. Additionally, the employer has lower profits due to this new higher salary that has to be paid to the Good type employee.